Schur and Power Sum Polytopes

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with

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Outline

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Hopf Algebras

A Hopf algebra (H, μ, Δ) is:

- An algebra. For instance $H := \bigoplus_{n \geq 0} \mathbb{K}\mathcal{S}_n$, with $m : H \otimes H \longrightarrow H$:

$$m(132 \otimes 21) = 13254 + 13524 + 13542 + 15324 + 15342 + 15432$$
$$= 51324 + 51342 + 51432 + 54132$$

- A coalgebra.

$$\Delta(13254) = 1 \otimes 13254 + 1 \otimes 2143 + 12 \otimes 132 + 132 \otimes 21 + 1324 \otimes 1 + 13254 \otimes 1$$

- A bialgebra. The comultiplication and counity maps are algebra maps.

Hopf Algebras

Definition

A Hopf algebra $(\mathbf{H}, m, \Delta, u, \epsilon)$ is a bialgebra with a linear map $S : \mathbf{H} \longrightarrow \mathbf{H}$ that is the inverse of the identity map $id_{\mathbf{H}}$ in the algebra $\mathrm{Hom}(\mathbf{H}, \mathbf{H})$.

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You should think of the antipode as a generalization of the Möbius function. Indeed, for $\mathbf{H} = \mathbb{K}\mathcal{P}$, and $\zeta(P) := 1$ for all $P \in \mathcal{P}$:

$$\mu = \zeta \circ S$$

Hopf Monoids

A Hopf monoid (F, μ, Δ) is:

- A monoid. For instance $S[I] := \{\text{Set partitions on } I\}$, with $\mu_{A,B} : F[A] \times F[B] \longrightarrow F[A \sqcup B]$:

$$\mu_{124,35}(124, 35) = 12435.$$

- A comonoid, with coproduct $\Delta_{A,B} : F[A \sqcup B] \longrightarrow F[A] \times F[B]$. We use the notation $\Delta_{A,B}(x) = (x|_A, x/_A)$.

$$\Delta_{13,245}(12534) = (13, 254).$$

- A bimonoid.

$$\Delta_{12,345}(12435) = (12, 435) = (124|_{1,2} \cdot 35|_{1,2}, 124/_{1,2} \cdot 35/_{1,2})$$

Outline

Standard Permutahedra

Definition

Take I an arbitrary set, with $n := |I|$. The *standard permutahedron* π_I is the convex hull of the set:

$$P := \{(\sigma(1), \ldots, \sigma(n)) \in \mathbb{R}^I \mid \sigma \in \mathcal{S}_n\}.$$

Where $\mathbb{R}^I := span(e_i)_{i \in I}$.

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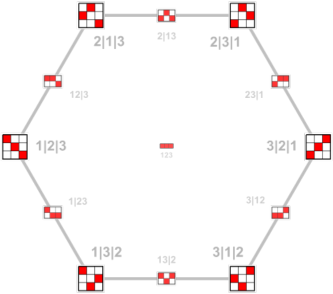
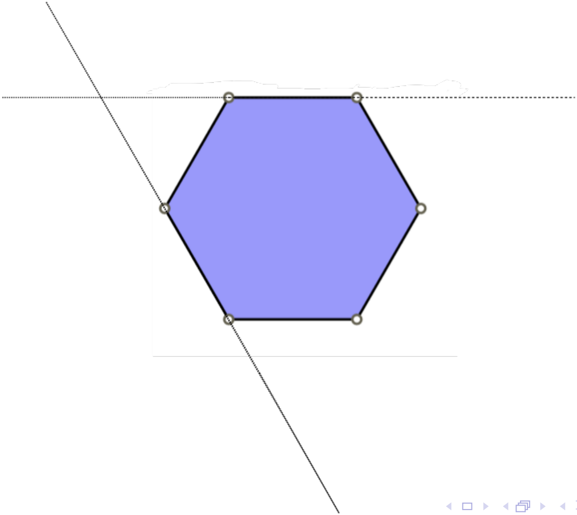


Generalized Permutahedra

A *generalized permutahedron* is a "deformation" of the standard permutahedron.



Generalized Permutahedra

- For any two generalized permutahedra $\mathfrak{p} \in GP[A]$, $\mathfrak{q} \in GP[B]$ one can define:

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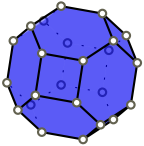
- The face $\mathfrak{p}_{S,T}$ optimized by the linear functional $1_S = \sum_{i \in S} x_i$, is a generalized permutahedron:

$$\mathfrak{p}_{S,T} = \mathcal{P}(z|_S) \times \mathcal{P}(z/_S)$$



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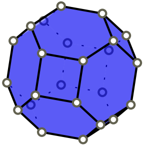

Theorem (Aguiar, Ardila 2017)

The species of generalized permutahedra GP, endowed with the product and coproduct described previously is a Hopf monoid.

Outline

Symmetric Functions

A *symmetric function* is a member of the ring $R[x_1, x_2, \ldots]$ of formal power series over countably infinite indeterminates, invariant under permutations of its subscripts.

For $n \geq 0$, we define:

- The *homogeneous symmetric function* as the symmetric function:

$$h_n := \sum_{i_1 \leq i_2 \leq \ldots \leq i_n} x_{i_1} \ldots x_{i_n}.$$

- The *elementary symmetric function*, as the symmetric function:

$$e_n := \sum_{i_1 < i_2 < \ldots < i_n} x_{i_1} \ldots x_{i_n}$$

- The *power sum symmetric functions* as the symmetric function:

$$p_n := \sum_i x_i^n$$

Symmetric Functions

For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, and $f \in \mathit{Sym}$, we let $f_\lambda = f_{(\lambda_1,\ldots,\lambda_k)}$ signify:

$$f_\lambda = f_{\lambda_1} \ldots f_{\lambda_k}.$$

Theorem

The symmetric functions $(h_\lambda)_\lambda$, (p_λ), and $(e_\lambda)_\lambda$ are all bases for *Sym* as a vector space.

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There is yet another basis for *Sym*, specially relevant due to its connection to the representation theory of the symmetric group; namely, that of *Schur functions*:

$$s_{\lambda'} := \left|e_{\lambda_i - i + j}\right|_{1 \le i,j \le k = len(\lambda)} = \begin{vmatrix} e_{\lambda_1} & e_{\lambda_1+1} & \cdots & e_{\lambda_1+k-1} \\ e_{\lambda_2-1} & e_{\lambda_2} & \cdots & e_{\lambda_2+k-2} \\ e_{\lambda_3-2} & e_{\lambda_3-1} & \cdots & e_{\lambda_3+k-3} \\ \vdots & \vdots & \vdots & \vdots \\ e_{\lambda_k-k+1} & e_{\lambda_k-k+2} & \cdots & e_{\lambda_k} \end{vmatrix}$$

Symmetric Functions

Example:

$$s_{(3,3,1)} = s_{(3,2,2)'} = \begin{vmatrix} e_3 & e_4 & e_5 \\ e_1 & e_2 & e_3 \\ e_0 & e_1 & e_2 \end{vmatrix} = e_3 {e_2}^2 - {e_3}^2 e_1 - e_4 e_1 e_2 + e_4 e_3 + e_5 {e_1}^2 - e_5 e_2.$$

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Motivation

- *Sym* has a Hopf algebraic structure given by the product and coproduct

$$\mu(f,g) := f \cdot g, \qquad \Delta(h_n) := \sum_{i=0}^{n} h_i \otimes h_{n-i}.$$

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- Let Π be the submonoid of *GP* generated by standard permutahedra.

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$$\Pi \twoheadrightarrow \overline{\Pi} \xrightarrow{\overline{\mathcal{K}}} Per \xrightarrow{\cong} Sym.$$

- *Per* and *Sym* are isomorphic. More precisely, through the morphism $\phi : Sym \longrightarrow Per$, defined by $\phi(n!h_n) = \pi_n$.

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- Can we arrive at compact descriptions of such objects? What does that tell us about the original symmetric function?
- Is there a way to see the Pieri rule, or change of basis formulas geometrically?
- At least for h_n, and e_n it is possible.

$$\phi(n!h_n) = \pi_n.$$

$$\phi(n!e_n) = (-1)^{n+1}\mathring{\pi}_n.$$

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Elementary Polytopes

- The antipode of GP is given by:

Theorem (Aguiar, Ardila 2017)

The antipode of the Hopf monoid **GP** of generalized permutahedra on $\mathfrak{p} \in \mathbf{GP}[I]$ is:

$$s_I(\mathfrak{p}) = (-1)^{|I|} \sum_{\substack{\mathfrak{f} \subseteq \mathfrak{p} \\ \mathfrak{f} \text{ is a face of } \mathfrak{p}}} (-1)^{dim(\mathfrak{f})} \mathfrak{f}.$$

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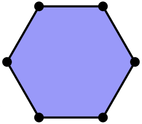
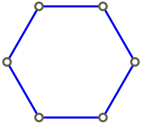
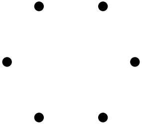
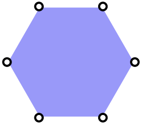
- Thus,

$$\phi((-1)^n n! e_n) = \phi(s(n! h_n)) = s(\phi(n! h_n)) = s(\pi_n) = -\mathring{\pi}.$$

Example

$s_{123}(\pi_3) = (-1)^3$  $+ (-1)^2$  $+ (-1)^1$ 

$= -$ 

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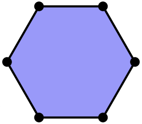
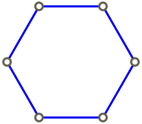
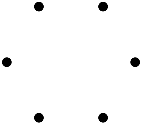
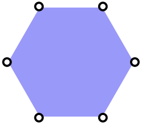
$s_{123}(\pi_3) = (-1)^3$  $+ (-1)^2$  $+ (-1)^1$ 

$= -$ 

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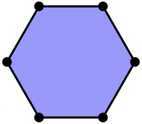
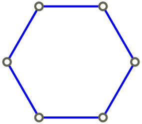
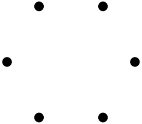
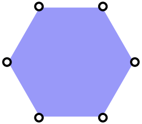
$s_{123}(\pi_3) = (-1)^3$  $+ (-1)^2$  $+ (-1)^1$ 

$= -$ 

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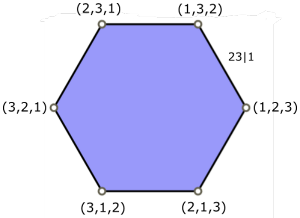

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- How can we associate in a uniform way an element of Per to a given symmetric function?

Definition

Let $f \in Sym_n$, and c be the coefficient of the expansion of f in the $(h_\lambda)_\lambda$ basis. The *polytope associated to that function* is $\phi(\frac{n!}{c}f) \in Per$, under the isomorphism $\phi : Sym \xrightarrow{\sim} Per$, if $c \neq 0$, and $\phi(n!f)$ if $c = 0$.

Elementary Polytopes

Theorem (Aguiar, Ardila 2017)

The Elementary Polytope $\mathcal{E}_n$ is the interior of the $(n-1)$-dimensional permutahedron up to a sign:

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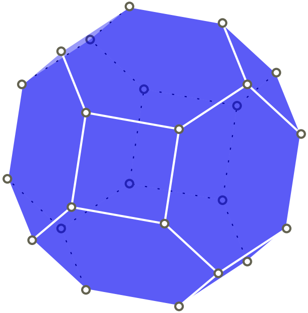

Elementary Polytopes

Symmetric Function	Associated Polytope
Homogeneous symmetric functions	The standard permutahedron π_n
Elementary symmetric functions	The standard permutahedron $(-1)^{n+1}\mathring{\pi}_n$
Power sum symmetric functions	?
Schur functions	?

Outline

Poset of set partitions

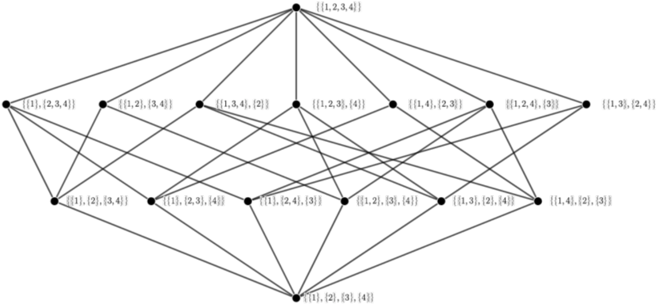

Figure: Hasse diagram of the partition lattice of 4 elements as seen on Formal approaches to a definition of agents. Biehl, Martin. (2017)

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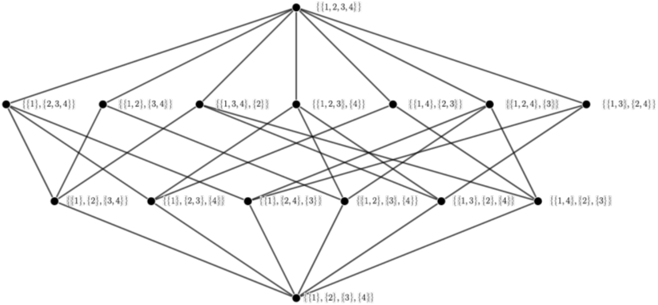

Figure: Hasse diagram of the partition lattice of 4 elements as seen on Formal approaches to a definition of agents. Biehl, Martin. (2017)

The Möbius function of this poset is known. It is given by:

$$\mu_*(0,\omega) = (-1)^{|\omega|}(|\omega|-1)!$$

Power Sums

- By the work of Doubilet,

$$p_n = \frac{1}{\mu_*(0,1)} \sum_{[n] \leq \omega} \mu_*(\omega, 1)(\omega_1 !) h_{\omega_1} \dots (\omega_k !) h_{\omega_k}$$

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Remark

Different set partitions with the same type are going to be taken into account twice in the index of the sum above.
Accordingly, there will be grouping of terms.

Power Sums

Lemma

Let ω be a given set partition. The number of faces of π_n with an associated set composition such that 1 belongs to the first part, and whose integer partition is equal to the integer partition affiliated to ω, is $|\mu_*(\omega, 1)| = (|\omega| - 1)!$ where $|\omega|$ is the number of parts of ω.

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$$\mathcal{P}_n = (-1)^{n-1} \sum_{[n] \leq \omega} \mu_*(\omega, 1) \pi_{\omega_1} \dots \pi_{\omega_k}$$

Remark

The facets showing in the expansion are half of all the facets of their type.

Power Sums

Theorem (Benedetti, E., Sanchez)

The Power Polytopes $\mathcal{P}_n$ are the whole permutahedron $(-1)^{n-1}\pi_n$ without half of its facets. Concretely, the permutahedron π_n up to a sign, without those of its facets with corresponding set composition S satisfying that 1 belongs to its first part.

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- Whenever $\mathfrak{p}$ and $\mathfrak{q}$ are faces with $\dim(\mathfrak{p}) = \dim(\mathfrak{q})$ their sign will be the same.
- It is enough to understand the behaviour of faces with 1 in their first part, in relation to the faces that lack it.

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Example:

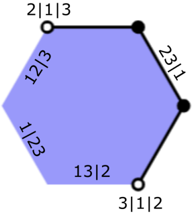

Figure: The Power Polytope $\mathcal{P}_3$

Outline

1. Preliminaries
   - Hopf Algebras and Hopf monoids
   - Generalized Permutahedra
   - Symmetric Functions
2. Elementary Polytopes
   - Motivation
   - Elementary Polytopes
3. Power Polytopes
   - Doubilet's Formula and Power Sums
4. Schur Polytopes
   - Winkel's Expansion
   - Schur Polytopes
5. Pieri Rule
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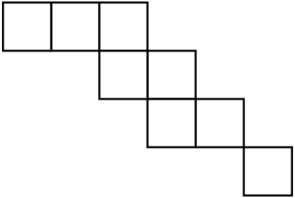

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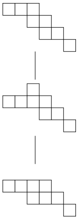


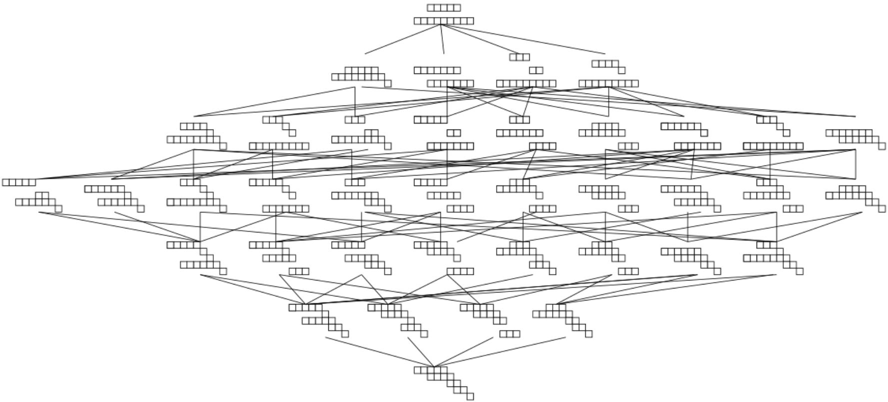
Definition

An admissible move between two staircase box diagrams D and D' is a move which transfers h boxes from row a in D to row b ($b > a$) in D', satisfying that for all $a \leq c \leq b$:

$$r(c) \geq r(a) + (c - a) \quad \text{or} \quad r(c) \leq r(a) - h + (c - a)$$

Where $r(c)$ is the number of boxes on row c.

The Poset D_λ



Winkel's Expansion

Theorem (Winkel, 1998)

Given a partition λ, and n the degree of its diagram D_λ, one has that the Schur function $s_{\lambda'}$ can be expressed as:

$$s_{\lambda'} = \sum_{D \in \mathcal{D}(\lambda)} (-1)^{\rho(D)} e_{r_D(1)} \cdots e_{r_D(n)}$$

Where ρ is the rank function of the poset with $\rho(D_\lambda) = 0$, and $r_D(i)$ the number of boxes of D in row i.
Also, $\mathcal{D}(\lambda)$ is isomorphic to a principal order ideal of the Bruhat order on $\mathcal{S}_l$, where l is the length of λ.

Winkel's Expansion

- For elementary symmetric functions instead of standard elementary monomials.

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- The non-trivial terms of the Jacobi-Trudi determinant are in correspondence with the diagrams of $\mathcal{D}(\lambda)$.

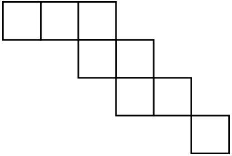

Figure: An admissible diagram corresponding to $e_3 e_2^2 e_1$

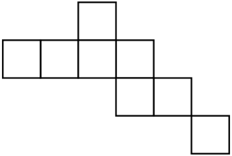

Figure: An admissible diagram corresponding to $e_1 e_4 e_2 e_1$

Outline

Schur Polytopes

According to our convention, the Schur polytope $\mathcal{S}_\lambda$ is given by:

$$\mathcal{S}_\lambda = n!\phi\left(\sum_{D\in\mathcal{D}(\lambda')}(-1)^{\rho(D)}e_{r_D(1)}\dots e_{r_D(n)}\right)$$

$$= n!\sum_{D\in\mathcal{D}(\lambda')}(-1)^{\rho(D)}\frac{\mathring{\pi}_{r_D(1)}}{r_D(1)!}\cdots\frac{\mathring{\pi}_{r_D(n)}}{r_D(n)!}.$$

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- Could it be that for each associated composition there is only one admissible diagram associated to it?
- Yes!

Schur Polytopes

Definition

Suppose that $M = (e_{\lambda_i - i + j})_{1 \leq i,j \leq n}$ is fixed, and let w be a word over the alphabet $(e_k)_{k \in \mathbb{N}}$. If w is the word of a permutation σ, then for all i, j:

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Lemma

Let $M = (e_{\lambda_i - i + j})_{1 \le i,j \le n}$ be fixed, $e_{\lambda_{\sigma(1)} - \sigma(1) + 1} \dots e_{\lambda_{\sigma(n)} - \sigma(n) + n}$ be a term of the determinant $|M|$ for some $\sigma \in \mathcal{S}_n$, and w be the *word of* σ. Then $e_{i'_1} \dots e_{i'_n}$ is a term of the determinant $|M|$ if and only if the word of the permutation $\omega(k) := i'_k$ satisfies:

$$\{k \mid \exists! j \in \mathbb{Z} \ : \ (w_\sigma)_k + j = (w_\omega)_{(k+j)}\} = [n]$$

Where w_k denotes the k-th entry of the word w.

Schur Polytopes

Theorem (Benedetti, E., Sanchez)

The Schur polytope $\mathcal{S}_\lambda$ is the polytope described by the expression:

$$\mathcal{S}_\lambda = \sum_{\substack{\mathcal{F} \leq \pi_n \\ \exists D:\ type(D) = type(\mathcal{F})}} (-1)^{asc(D) + dim(\mathcal{F})} \mathring{\mathcal{F}}$$

Where the sum is over the faces $\mathcal{F}$ such that there exists an admissible diagram D with the said condition.

Examples

According to the previous theorem, the Schur polytope $\mathcal{S}_\lambda$ for $\lambda = (2,1)$ is:

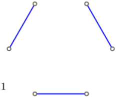
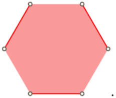
$\mathcal{S}_\lambda = (-1)^3$  $+ (-1)^1$ 

Diagram		
Ascents	1	0
Dimension	2	1

So that:

$\mathcal{S}_\lambda =$  .

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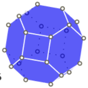
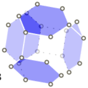
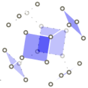
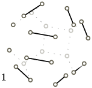
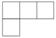
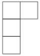
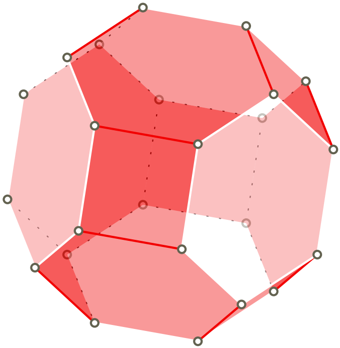
$\mathcal{S}_\lambda = (-1)^5$  $+(-1)^3$  $+(-1)^{2+1}$  $+(-1)^1$ 

				*Diag.*
2	1	1	0	*Asc.*
3	2	2	1	*Dim.*

Examples

$$\mathcal{S}_\lambda = $$  $.$

Examples

Example

Let λ be a hook partition $\lambda = (m, 1, \dots, 1)$. For any such partition the poset $\mathcal{D}(\lambda)$ has a remarkably simple form, namely, it is isomorphic to the filter generated by the set compositions whose integer composition corresponds to λ, in the poset SC.
Geometrically, this means that $\mathcal{S}_\lambda$ is the polytope that has as faces all the faces of the permutahedron of the form $\pi_m \times \pi_1 \times \dots \times \pi_1$, as well as all the faces that contain them.
Moreover, all those faces show up with sign -1.

Outline

The Littlewood-Richardson Coefficients

- Since the Schur functions are a basis for the symmetric functions, we can ask how to expand the product of Schur functions in that basis.

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- The coefficients $c^{\lambda}_{\mu,\nu}$ of that expansion are known as the *Littlewood-Richardson* coefficients.

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$$s_{\mu}s_{\nu} = \sum_{\lambda \vdash |\mu|+|\nu|} c^{\lambda}_{\mu,\nu} s_{\lambda}.$$

- With the Hopf algebraic structure defined before, the coproduct of an arbitrary Schur function in *Sym* can be given with the aid of the Littlewood-Richardson coefficients:

$$\Delta(s_{\lambda}) = \sum_{\mu,\nu \,:\, |\mu|+|\nu|=|\lambda|} c^{\lambda}_{\mu,\nu}(s_{\mu} \otimes s_{\nu}).$$

The Pieri Rule

- In the case when $\mu = (n)$, i.e. when one of the partitions indexing the product $s_\mu s_\nu$ has only one part, there is an easy description of that expansion (equivalently of the Littlewood-Richardson coefficients).

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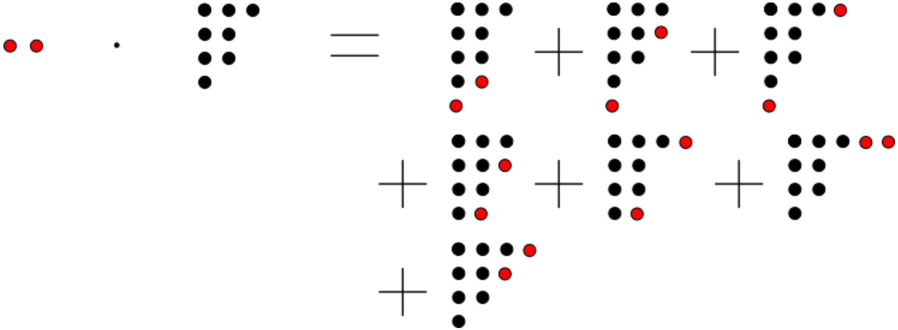
- That description is known as the *Pieri rule*.

Theorem (Pieri Rule)

The product of the Schur functions $s_{(n)} s_\mu$ is the sum of those Schur functions s_λ such that the Ferrer diagram of λ can be obtained by adding n boxes to the Ferrer diagram of μ; in such a way that no two boxes are in the same column.

Example

For the partitions $\mu = (2)$ and $\nu = (3, 2, 2, 1)$, the product of the Schur functions $s_\mu \cdot s_\nu$ is the sum of the Schur functions s_λ, such that λ is any of the partitions on the right hand side of the equation below:



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- At the same time,

$$\Delta(\mathcal{S}_\lambda) = \sum_{S \sqcup T = I:\ |I|=n} \Delta_{S,T}(\mathcal{S}_\lambda).$$

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$$c^{\lambda}_{(1),\nu} = 1 \iff [\mathcal{S}_{(1)} \otimes \mathcal{S}_\nu](\Delta(\mathcal{S}_\lambda)) \neq 0.$$

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- There is a geometric way to find out if $\Delta_{S,T}(\mathfrak{p}) \neq 0$!

Example

- Suppose we want to know the expansion of $s_{(1)} \cdot s_{(2,1)} = h_1 \cdot s_{(2,1)}$ in the Schur basis:

$$s_{(1)} \cdot s_{(2,1)} = \sum_{\lambda} c^{\lambda}_{(1),(2,1)} s_{\lambda}.$$

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- Take $\lambda = (3,1)$. The drawing below shows that $c^{(3,1)}_{(1),(2,1)} = 1$.

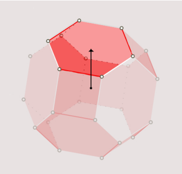

Figure: The Schur polytope $\mathcal{S}_{(3,1)}$

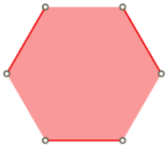

Figure: The Schur polytope $\mathcal{S}_{(2,1)}$

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Theorem (Aguiar, Ardila 2017)

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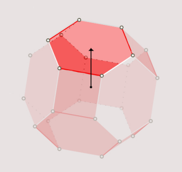
- Thus, we can calculate the coproduct in the poset of set partitions.

Pieri Rule in Higher Dimensions

- All the faces of a Schur polytope are just elements of the poset of admisible diagrams thought of as tabloids.



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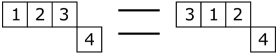


Figure: The optimized face as a tabloid.

Pieri Rule in Higher Dimensions

- The coproduct is easy to understand on the faces (tabloids).

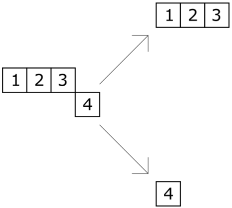


Figure: The action of $\Delta_{123,4}$ on a tabloid with 4 apart.

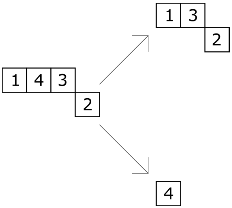
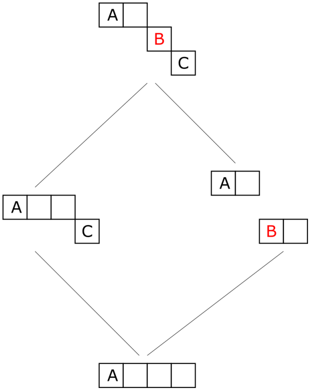


Figure: The action of $\Delta_{123,4}$ otherwise.

Pieri Rule in Higher Dimensions

- The way in which the coproduct acts on tabloids induces a labelling on them.



Pieri Rule in Higher Dimensions

Theorem (Benedetti, E., Sanchez)

Let $\lambda \vdash n$ be a partition. Then, there is a labelling of the poset S of set compositions of $[n] := \{1, \dots, n\}$ such that:

1. The set of faces of $\mathcal{S}_\lambda$ is a subposet P.
2. There is a sign reversing involution ϕ on the filter generated by D_{λ^-} within P; with λ^- a diagram obtained by removing a block from λ, and λ^- not a partition.
3. The subposet P contains the set of faces of $\mathcal{S}_{\lambda^-}$ for all the partitions λ^- so that λ^- a diagram obtained by removing a block from λ.
4. All of the elements of P are either of the form of 2 or 3.